Geometric Algebra: A natural representation of three-space

James M. Chappell*

(a) School of Electrical and Electronic Engineering,

(b) School of Chemistry and Physics,

University of Adelaide, SA 5005, Australia

Azhar Iqbal

a) School of Electrical and Electronic Engineering,

University of Adelaide, SA 5005, Australia

b) Centre for Advanced Mathematics and Physics,

National University of Sciences & Technology, Sector H-12, Islamabad, Pakistan

Derek Abbott

School of Electrical and Electronic Engineering,

University of Adelaide, SA 5005, Australia

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Abstract

Historically, there have been many attempts to define the correct algebra for modeling the properties of three dimensional physical space, such as Descartes' system of Cartesian coordinates in 1637, the quaternions of Hamilton representing rotation in three-space that built on the Argand diagram for two-space, and Gibbs' vector calculus employing the dot and cross products. We illustrate however, that Clifford's geometric algebra developed in 1873, but largely overlooked by the science community, provides the simplest and most natural algebra for three-space and hence has general applicability to all fields of science and engineering. To support this thesis, we firstly show how geometric algebra naturally produces all the properties of complex numbers and quaternions and the vector cross product in a single formalism, whilst still maintaining a strictly real field and secondly we show in two specific cases how it simplifies analysis in regards to electromagnetism and Dirac's equation of quantum mechanics. This approach thus has the immediate advantage of removing complex fields from analysis, so that algebraic entities have a geometric expression in real space. As an example, we show how quadratic equations now can be given two real solutions using GA. This viewpoint also has something interesting to say about the concept of number itself, because numbers in GA can encapsulate scalars, lines, areas and volumes, into a single entity, which can be multiplied and divided, just like normal numbers. This simple and elegant formalism would also seem very appropriate for students learning vectors, algebra and geometry, giving a more natural and intuitive understanding of the properties of three-space.

*Electronic address: james.m.chappell@adelaide.edu.au

I. INTRODUCTION

Einstein once stated, 'Everything should be made as simple as possible, but not one bit simpler', and in this paper we ask the question: 'What is the simplest and most natural algebraic representation of three space?'

It is readily accepted from experience and scientific experiment that we have three spatial dimensions or independent directions of movement within which to describe events, and hence Cartesian coordinates seem eminently appropriate for the task of providing a framework for three-space, where we typically label the three orthogonal directions as x, y, z. We ignore in the first instance the presence of curved space, or other minute curled up dimensions. For three-space, though, as well as positional coordinates, we also need to be able to represent orientation or rotations at each point in the space. In two-space the correct algebra is defined by complex numbers and for three space, the algebra for rotations is given by Hamilton's quaternions. Quaternions are seen as somewhat unusual due to their non-commuting properties, however this reflects the observed non-commutivity of rotations in three-space. This can very easily be demonstrated by rotating any object, with a $90°$ and $180°$ orthogonal rotation, where one will then find the relation that $R_{180}R_{90} = -R_{90}R_{180}$. Hence, in order to form a unified algebra of three-space we need to integrate the complex numbers and quaternions within the framework of Cartesian coordinates. This was achieved by Clifford in 1873, who named his system, *Geometric Algebra*, or GA.

As well as deriving Clifford's system, we also illustrate, how the formalism indeed replaces, complex numbers, quaternions, the vector cross product, rotation matrices, the Pauli matrices and the anti-symmetric field tensor as well as several other formalisms. Clearly this simplification should have general applicability to many fields of science, physics, and engineering as well as being advantageous in an educational setting.

A. Historical development

Around 547 BCE, Thales expressed his belief that every event on the earth had natural rather than mythological causes. For example he suggested that earthquakes were not caused by the machinations of the gods on Mount Olympus, but proposed instead that the earth floated on water and was being buffeted by large waves [1]. Considering the limited

knowledge of the time, this showed very good insight, as we would perhaps refine this today by saying that the earth floats on molten magma and is buffeted by tectonic plates. For this reason Thales is sometimes identified as the first scientist. Pythagoras extended this notion of logical causation behind events, postulating that numbers and their relationships, or mathematics, underly all things. This idea was then masterfully applied by Euclid to geometry, deriving a multitude of geometrical theorems from a few simple axioms. Euclid's books became standard texts for the next 2000 years, only disappearing from school curricula in the 20th century!

The next major development came in 1637 when Descartes unified algebra and geometry onto the Cartesian plane, so that the three independent spatial freedoms, could be unambiguously represented by a single set (vector) holding the three coordinates. This achievement is called by John Stuart Mill, 'the greatest single step ever made in the exact sciences' [2]. It has been extensively debated by historians, however, why there was such a slow down in the progress of science and mathematics for over 1500 years following the Greek explosion. Various suggestions have been provided to answer this, such as the Roman empire suppressing dissent and not sponsoring the arts, the rise of the roman catholic church, the ready availability of slaves obviating the need for work efficiencies and the need to educate them and so on [3]. However Hestenes has put forward another possibility, that the algebraic and numerical system used by the Greeks, had inherent limitations, which were roadblocks to further progress [4].

For example the length along the diagonal of a unit square, we know today as $\sqrt{2}$, being an irrational number, did not exist in the Greek numeric system, which was based solely on integers and their ratios. Another hindrance would have been the roman numerals which made numeric manipulation difficult. Also the algebraic expression x^3, or x cubed, referred to a physical cube and hence x^4 was unacceptable on idealogical grounds. However with the arrival of Hindu-Arabic numbers in about 1000 AD into Europe, which included a zero, thus allowing positional representation for numbers in base 10, together with the acceptance of negative numbers in 1545 AD, allowing for the concept of a complete number line, paved the way for Descartes to revolutionize the Greek system, by proposing a union of algebra and geometry using Cartesian coordinates. He stated 'Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division and the extraction of roots, which may be considered a kind of division, so in geometry, to find required lines it is

merely necessary to add or subtract lines.' Descartes thus postulated an equivalence between line segments and numbers, something the Greeks were not prepared to do. Effectively he generalized the concept of number to include directed line segments. We will show in fact, that the concept of number can be further extended beyond line segments, to include planes and volumes.

This simple approach of representing two-space and three-space by Descartes, however, becomes confused with the development of the Argand diagram, which also represents the plane, but consisting of one real and one imaginary coordinate, and to make matters worse, Hamilton in 1843 generalized complex numbers to three space, naming them quaternions. This confused state of affairs, on exactly how to represent three-space coordinates and rotations, was finally resolved by William Clifford in 1873, who combined Cartesian coordinates and the rotational algebra of both complex numbers and quaternions into single unified algebraic framework.

However Clifford also achieved a fulfillment of Descartes' original vision of a vector being able to be manipulated in the same way as normal numbers, by deriving the appropriate multiplication and division operations for vectors, which meant that geometric objects such as vectors could now treated with normal algebraic operations. Cliffords mathematical system 'should have gone on to dominate mathematical physics' [5], but, Clifford died young, at the age of just 33 and vector calculus was heavily promoted by Gibbs and rapidly became popular, eclipsing Clifford's work, which in comparison appeared strange with its non-commuting variables.

Hestenes has championed GA for over forty years and the development of GA is now expanding rapidly, with research in black holes [6], quantum field theory [7], quantum tunneling [8], quantum computing [9], general relativity and cosmology [10], beam dynamics and buckling [11], computer vision [12] and EPR-Bell experiments [13].

B. Clifford's definition of three-space

How did Clifford solve the problem of the correct representation for three-space, integrating the rotational algebra of complex numbers and quaternions? In order to describe Clifford's approach, we firstly define the three Cartesian coordinate directions by e_1, e_2 and e_3 as shown in Fig 1, as proposed by Descartes. We specify the orthonormality of these

three vectors with the expression $e_i \cdot e_j = \delta_{ij}$, where the conventional vector dot product is adopted without alteration in GA, however Clifford then defined each algebraic element e_i as anticommuting, such that $e_i e_j = -e_j e_i$. The bivectors e_2e_3, e_1e_3 and e_2e_3 square to minus one, for example, $(e_2e_3)^2 = e_2e_3e_2e_3 = -e_2e_2e_3e_3 = -1$, where $e_i^2 = 1$, and the bivectors now become isomorphic to the quaternions of Hamilton. The trivector $\iota = e_1e_2e_3$ also squares to minus one and commutes with the three orthonormal basis vectors and hence is isomorphic to the imaginary number $\sqrt{-1}$. Note that the trivector ι has identical properties, but is not the same object as the imaginary number $\sqrt{-1}$, normally represented with the symbol i or j, so this is indicated by giving it a slightly different symbol ι, the Greek iota symbol.

We can also find $e_ie_j = \iota\epsilon_{ijk}e_k$, which we call the dual representation, which we see creates the Pauli algebra, and hence we can use Clifford's basis vectors to replace the Pauli matrices, as they also square to one and anticommute.

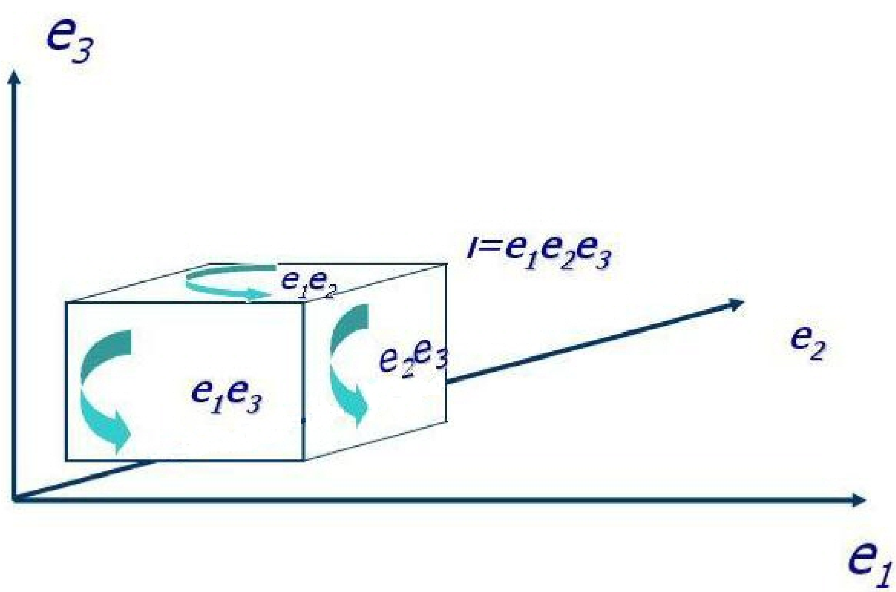

FIG. 1: Clifford's model for three space.

Thus far, using Clifford's GA, we have already superseded, complex numbers, quaternions and the Pauli algebra, into a completely real three-space algebra. We can now also explain, the somewhat mysterious imaginary number $i = \sqrt{-1}$, as simply an algebraic stand in for two-space rotations, or the trivector ι, also an algebraic object that squares to minus one. Also, quaternions, ironically, usually seen by students as some frightening beast locked in the mathematics basement, should be seen as the most natural of operations, being simply

three-space rotations represented by the bivectors $\iota e_1, \iota e_2, \iota e_3$ in GA.

C. Derivation of the Clifford product

Clifford also defined the geometric product, using simply the distributive law for expanding brackets. Hence the product of two vectors becomes

$$
\begin{aligned}
& \mathbf{uv} \qquad (1)\\
&= (e_1u_1 + e_2u_2 + e_3u_3)(e_1v_1 + e_2v_2 + e_3v_3)\\
&= u_1v_1 + u_2v_2 + u_3v_3 + (u_2v_3 - v_2u_3)e_2e_3 + (u_1v_3 - u_3v_1)e_1e_3 + (u_1v_2 - v_1u_2)e_1e_2\\
&= u_1v_1 + u_2v_2 + u_3v_3 + \iota\left((u_2v_3 - v_2u_3)e_1 + (u_1v_3 - u_3v_1)e_2 + (u_1v_2 - v_1u_2)e_3\right)\\
&= \mathbf{u}\cdot\mathbf{v} + \iota\mathbf{u}\times\mathbf{v},\\
&= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v},
\end{aligned}
$$

using $e_i^2 = 1$ and $e_ie_j = -e_je_i$, which thus provides a definition of the dot and outer products in component form. Hence Clifford's vector product naturally integrates with common algebraic operations, being simply the algebraic product of two brackets. We can also see that for the case of a vector times itself, that the cross product will be zero and hence the square of a vector $\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v}$, becomes a scalar quantity.

Note that the expression $\mathbf{u}\cdot\mathbf{v}+\iota\mathbf{u}\times\mathbf{v}$, is in the form of a complex-like number, and in the same way as we prefer to leave a normal complex number $a + ib$ as a single unit rather than separating it into real and imaginary parts, it is also preferable to leave $\mathbf{u}\cdot\mathbf{v} + \iota\mathbf{u}\times\mathbf{v}$ as a whole rather than splitting it into the separate dot and cross products. This, in fact, appears to be a foundational error of vector analysis, as the separate dot and cross products, now fail to integrate with normal algebraic operations, making it more difficult to carry out algebraic manipulation, and hence, also harder for students to master. However if we leave the Clifford vector product intact one finds that GA can naturally be employed to represent quantum mechanics in real three-space coordinates without the use of Pauli or Dirac matrices [14].

The cross product as defined by Gibbs also has a problem, in that it only applies in three dimensional space, because in four dimensions there is an infinity of perpendicular vectors, and so it is preferable to adopt the wedge product in its stead, which applies generally in

any number of dimensions, as does the dot product. Historically, as Gibbs' system of vectors became more popular in physics and in various other fields, new scientific discoveries such as quantum mechanics and relativity meant that vector analysis needed to be supplemented by many other mathematical techniques such as: tensors, spinors, matrix algebra, Hilbert spaces, differential forms etc. and as noted in [15], 'The result is a bewildering plethora of mathematical techniques which require much learning and teaching, which tend to fragment the subject and which embody wasteful overlaps and requirements of translation.'

D. Maxwell's equations in GA

Electromagnetism is one of the foundational theories of physics and we find significant simplifications if GA is adopted. Maxwell's equations were first published in 1865 [16] mathematically describing then recently discovered electromagnetic phenomena. His equations were written for three-space, requiring 12 equations in 12 unknowns. These equations were later rewritten by Heaviside and Gibbs, in the newly developed formalism of dot and cross products, which reduced them to the four equations now seen in most modern textbooks [17] and shown below in S.I. units

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon}, \quad \text{(Gauss' law);} \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0, \quad \text{(Faraday's law);} \\ \nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} &= \mu_0 \mathbf{J}, \quad \text{(Ampère's law);} \\ \nabla \cdot \mathbf{B} &= 0, \quad \text{(Gauss' law of magnetism),} \end{aligned} \tag{2}$$

where $\mathbf{E}, \mathbf{B}, \mathbf{J}$ are conventional vector fields, with $\mathbf{E}$ the electric field strength and $\mathbf{B}$ the magnetic field strength and ∇ is the three-vector gradient given by

$$\nabla = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z}, \tag{3}$$

and e_1, e_2, e_3 are the three Euclidean space orthonormal vectors, with $e_i \cdot e_j = \delta_{ij}$, where δ_{ij} is the Kronecker-delta symbol. Maxwell's four equations given in Eq. (2) along with the Lorentz force law

$$\mathbf{K} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{4}$$

completely summarize classical electrodynamics [17]. It can be seen that we have adopted the convention that a three-space vector $\mathbf{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$, is identified in bold, in agreement with ISO 80000-2 international standard for mathematical notation.

However inspecting the form of the geometric product for $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \iota \mathbf{u} \times \mathbf{v}$, we can see that these equations can be fairly easily combined. If we firstly multiply the second and fourth equations by ι and insert ι into Eq. (5) as shown

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon}, \quad \text{(Gauss' law)}; \\ \iota\nabla \times \mathbf{E} + \partial_t \iota \mathbf{B} &= 0, \quad \text{(Faraday's law)}; \\ \iota\nabla \times \iota\mathbf{B} + \frac{1}{c^2}\partial_t \mathbf{E} &= -\mu_0 \mathbf{J}, \quad \text{(Ampère's law)}; \\ \nabla \cdot \iota\mathbf{B} &= 0, \quad \text{(Gauss' law of magnetism)}. \end{aligned} \tag{5}$$

We now see that using Eq. (1) that we can add the first and second and the third and fourth equations to find

$$\begin{aligned} \nabla\mathbf{E} + \partial_t \iota \mathbf{B} &= \frac{\rho}{\epsilon}; \\ \nabla\iota\mathbf{B} + \frac{1}{c^2}\partial_t \mathbf{E} &= -\mu_0 \mathbf{J}, \end{aligned} \tag{6}$$

where $\nabla\mathbf{E}$ will now represent the geometric product of two vectors, that is $\nabla\mathbf{E} = \nabla \cdot \mathbf{E} + \iota\nabla \times \mathbf{E}$. However, these two remaining equations can now be added to produce

$$(\frac{1}{c}\partial_t + \nabla)(\mathbf{E} + \iota c\mathbf{B}) = \frac{\rho}{\epsilon} - c\mu_0 \mathbf{J}. \tag{7}$$

If we define $F = \mathbf{E} + \iota c\mathbf{B}$ and the four-gradient $\Box = \frac{1}{c}\partial_t + \nabla$, with the source $J = \frac{\rho}{\epsilon} - c\mu_0\mathbf{J}$, we find

$$\Box F = J. \tag{8}$$

Thus unifying the dot and cross products, naturally reduces Maxwell's four equations down into a single equation, demonstrating the power of Clifford's approach. We can see that the $\mathbf{B}$ field, is written as a pseudovector $\iota\mathbf{B}$, as part of the field F. This is appropriate, because under a reflection, the current in a loop of wire will flow in the opposite direction, thus inverting the $\mathbf{B}$ field, whereas an $\mathbf{E}$ field is simply reflected about the mirror plane. This

different behavior of the **E** and **B** fields is obvious from the GA formalism but hidden in the tensor and Gibbs' vector formalism.

Also if we wished to describe Maxwell's original four equations in English we would find it fairly difficult. For example, we could describe Maxwell's equations as follows: equate the divergence of the **E** field to the charge present, simultaneously equating the Curl of **E** as the negative rate of change with time of the **B** field, simultaneously, and so on for the two remaining Maxwell equations shown in Eq. (2). However with GA we can simply say that the four-gradient of the field F is proportional to the amount of charge and current present.

E. The versatile multivector

As previously shown, the geometric product, is simply an algebraic expansion of two brackets, hence there is now no reason why we cannot combine vectors with scalars, bivectors or trivectors.

For example, adding all these components together, we produce a multivector, which can be written

$$M = a + \mathbf{v} + \iota\mathbf{w} + \iota b, \tag{9}$$

which shows in sequence, scalar, vector, bivector and trivector terms. This general three-space multivector can be used to represent a diverse range of mathematical objects. Firstly, we can obviously map into scalars (a), vectors ($\mathbf{v}$) and complex-type numbers ($a + \iota b$). The quaternions i, j, k, defined by $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$, map into the multivector $\iota\mathbf{w}$, that is, component wise, $\mathrm{i} \to \iota e_1, \mathrm{j} \to -\iota e_2, \mathrm{k} \to \iota e_3$. Hence a general quaternion is mapped to GA as

$$q = a + b\mathrm{i} - c\mathrm{j} + d\mathrm{k} \leftrightarrow a + \iota b e_1 + \iota c e_2 + \iota d e_3 = a + \iota\mathbf{w}. \tag{10}$$

This is similar to the Pauli spinor mapping

$$|\psi\rangle = \begin{bmatrix} a + ja_3 \\ -a_2 + ja_1 \end{bmatrix} \leftrightarrow \psi = a + \iota a_1 e_1 + \iota a_2 e_2 + \iota a_3 e_3 = a + \iota\mathbf{w}, \tag{11}$$

also mapping to the even sub algebra of the multivector, which shows the close relationship between spinors and quaternions [18].

The electromagnetic field represented by the antisymmetric tensor $F^{\mu\nu}$ [17], maps as

follows

$$F^{\mu\nu} \leftrightarrow F = \mathbf{E} + \iota c\mathbf{B}, \tag{12}$$

with the dual tensor $G^{\mu\nu}$ given in GA by $G = \iota F$. We also have other dual spaces in the multivector, such as for vectors $\mathbf{v} \leftrightarrow \iota\mathbf{w}$ and for spinors $a + \iota\mathbf{w} \leftrightarrow \iota(b - \iota\mathbf{v})$.

The Dirac wave function, used in the Dirac equation, maps to the full multivector as follows

$$|\psi\rangle = \begin{bmatrix} a + ja_3 \\ a_2 + ja_1 \\ -b_3 + jb \\ -b_1 + jb_2 \end{bmatrix} \leftrightarrow a_0 + b_1e_1 + b_2e_2 + b_3e_3 + \iota a_1e_1 + \iota a_2e_2 + \iota a_3e_3 + \iota b = a + \mathbf{v} + \iota\mathbf{w} + \iota b. \tag{13}$$

Other entities can also be mapped to a multivector, such as pseudoscalars (ιb) and pseudo vectors ($\iota\mathbf{w}$). Rotation matrices are also represented in a multivector as $R = a + \iota\mathbf{w}$, which rotates a vector about the $\mathbf{w}$ axis by an angle given by $\tan\theta = |w|/a$ according to

$$\mathbf{v}' = R\mathbf{v}R^{\dagger}, \tag{14}$$

where $R^{\dagger} = e_0\tilde{R}e_0$, where the tilde operation reverses the order of products and e_0 represents the time coordinate, which anticommutes with the space coordinates e_i. Hence the great versatility of the three-space multivector is demonstrated, being able to replace a large variety of mathematical structures and formalisms as shown and hence unify the mathematical language of physics [14].

F. Common algebraic operations on a multivector

Descartes claimed that five common algebraic operations, addition, subtraction, multiplication, division and the square root, could be applied to his line segments, however this idea can be extended to full multivectors, which contain not only line segments (vectors) but also areas (bivectors) and volumes (trivectors). Addition and subtraction are simply defined by adding like components, that is, if $M_1 = a_1 + \mathbf{v_1} + \iota\mathbf{w_1} + \iota b_1$ and $M_2 = a_2 + \mathbf{v_2} + \iota\mathbf{w_2} + \iota b_2$, then

$$M_1 + M_2 = (a_1 + a_2) + (\mathbf{v_1} + \mathbf{v_2}) + \iota(\mathbf{w_1} + \mathbf{w_2}) + \iota(b_1 + b_2), \tag{15}$$

and similarly for subtraction.

1. *Multiplication of multivectors*

The multiplication operation is given by an algebraic product, similar to the algebraic product of two vectors, that is

$$\begin{aligned} M_1M_2 &= (a_1 + \mathbf{v_1} + \iota\mathbf{w_1} + \iota b_1)(a_2 + \mathbf{v_2} + \iota\mathbf{w_2} + \iota b_2) \\ &= \big(a_1a_2 + \mathbf{v}_1 \cdot \mathbf{v}_2 - \mathbf{w}_1 \cdot \mathbf{w}_2 - b_1b_2 + a_2\mathbf{v_1} + a_1\mathbf{v_2} - b_2\mathbf{w_1} - b_1\mathbf{w_2} + v_1 \times w_2 + w_1 \times v_2 \\ &+ \ \iota(a_2\mathbf{w_1} + a_1\mathbf{w_2} + b_2\mathbf{v_1} + b_1\mathbf{v_2} + v_1 \times v_2 - w_1 \times w_2) + \iota(a_1b_2 + a_2b_1 + v_1 \cdot w_2 + w_1 \cdot v_2)\big), \end{aligned} \tag{16}$$

where we have used repeatedly the geometric product defined in Eq. (1).

2. *The inverse of a multivector*

For the division operation, we firstly need an inverse operation for a general multivector. For the general multivector defined in Eq. (9), we find the inverse

$$M^{-1} = (a - \mathbf{v} - \iota\mathbf{w} + \iota b)(f + \iota g)/(f^2 + g^2) \tag{17}$$

where we have the scalars $f = a^2 - v^2 + w^2 - b^2$ and $g = 2(\mathbf{v} \cdot \mathbf{w} - ab)$.

The special case of the inverse of a vector is given by

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{v^2}, \tag{18}$$

which can be checked through $\mathbf{v}\mathbf{v}^{-1} = \mathbf{v}\frac{\mathbf{v}}{v^2} = \frac{v^2}{v^2} = 1$ as required. Hence, we can see that the inverse of any vector is a vector in the same direction with the reciprocal of the length of the original vector.

3. *The square root of a multivector*

For the multivector defined in Eq. (9), we find the square root

$$M^{\frac{1}{2}} = \pm(c + \mathbf{x} + \iota\mathbf{y} + \iota d)\,(f + \iota g)\,, \tag{19}$$

where we find $c = r\cosh\frac{\theta}{2}$ and $d = r\sinh\frac{\theta}{2}$, where $\mathbf{x} = \frac{d\mathbf{n}+c\mathbf{m}}{2(c^2+d^2)}$ and $\mathbf{y} = \frac{c\mathbf{n}-d\mathbf{m}}{2(c^2+d^2)}$ and $f + \iota g = \frac{1}{\sqrt{2}}\left(\sqrt{a+\sqrt{h}} + \iota\sqrt{-a+\sqrt{h}}\right)$, with $h = a^2 + b^2$, using the following support

formulae

$$
\begin{aligned}
r &= \pm\left(\frac{q - p\,\mathrm{cosech}\theta}{\cosh^2\theta}\right)^{\frac{1}{4}} \quad , \quad \theta = \mathrm{arcsinh}\frac{p(1+4q+s)\pm u}{4(q-p^2)} \qquad (20)\\
q &= \frac{1}{4}(x^2-y^2) \ , \ p = \frac{1}{2}\mathbf{x}\cdot\mathbf{y}\\
s &= \sqrt{(4q-1)^2+16p^2} \ , \ u = \sqrt{2}\sqrt{16q^3+12qp^2+p^2-4q^2+s(p^2+4q^2)}\\
\mathbf{m} &= \frac{a\mathbf{v}+b\mathbf{w}}{a^2+b^2} \quad , \quad \mathbf{n} = \frac{a\mathbf{w}-b\mathbf{v}}{a^2+b^2}.
\end{aligned}
\qquad (21)
$$

Because multivector multiplication is associative we can now find all the rational powers $M^{k/2^t}$, where k, t are integers, which can approximate to any fractional power. We find that the square root generally exists, and in some cases there is a double root (ignoring signs).

G. Interpreting solutions of quadratics using GA

Imaginary numbers caused a lot of confusion when first discovered as solutions to quadratic equations, however with GA the imaginary solutions $i = \sqrt{-1}$, are given a geometric meaning using $\iota_2 = e_1e_2$, the bivector of two-space, which also squares to minus one. Hence a complex number solution $x = a + ib$ can be written in GA as $R = r\mathrm{e}^{\iota_2\theta} = r(\cos\theta + \iota\sin\theta)$, a rotor in the plane described using just real numbers. In two-space we can rotate vectors using $\mathbf{v}' = R\mathbf{v}$. That is if $R = \mathrm{e}^{\iota_2\pi/2} = \iota_2 = e_1e_2$, then if $\mathbf{v} = e_2$, then $\mathbf{v}' = e_1e_2e_2 = e_1$, or a clockwise rotation by ninety degrees. Hence solutions of quadratics using complex numbers, imply we are using rotations in the plane, instead of simply scaling along the real number line, that is we are increasing the solution space from a line to a plane. This also explains why we always have two symmetrical complex solutions, if they exist, as they represent $\pm\phi$ directions for the rotor. Then, if we write the quadratic equation as

$$-ax^2 - bx = c \qquad (22)$$

and because x now can become a rotor, we can view this as a rotor equation, where the action of the LHS, which involves rotating the starting vector is equivalent to a pure scaling of a vector by c. that is, if $Q_1 = -ax^2 - bx$ and $Q_2 = c$, then

$$Q_1\mathbf{v} = Q_2\mathbf{v}, \qquad (23)$$

where the real number line is assumed to be the e_1 axis, so that $\mathbf{v} = e_1$. We can see that the rotor operator Q_2 simply scales the initial vector by c. Hence we require the action of the operator Q_1, which allows rotation of the vector in the plane, to return the vector to the starting direction and scale it by c. Hence, thinking in terms of adding vectors head to tail, then adding the two vectors $-ax^2\mathbf{v}$ and $-bx\mathbf{v}$, will produce a vector equal to the vector $c\mathbf{v}$.

That is

$$\begin{aligned} Q_1\mathbf{v} &= (-aR^2 - bR)\mathbf{v} \\ &= (-ar^2\mathrm{e}^{2\iota_2\theta} - rb\mathrm{e}^{\iota_2\theta})\mathbf{v} \\ &= c\mathbf{v} \end{aligned} \tag{24}$$

where we used the property of exponentials that $(\mathrm{e}^{\iota_2\theta})^2 = \mathrm{e}^{2\iota_2\theta}$.

Firstly we see that the e_2-component of the vector transformed by Q_1 must be zero, which ensures the vector is parallel to the starting vector, hence we have the equation

$$ar^2 \sin 2\theta + rb \sin\theta = 0, \tag{25}$$

giving $\cos\theta = -\frac{b}{2ar}$. For the e_1 direction we have

$$\begin{aligned} -ar^2\cos 2\theta - rb\cos\theta &= c \\ -ar^2(2\cos^2\theta - 1) - br\cos\theta &= c \\ -\frac{b^2}{2a} + ar^2 + \frac{b^2}{2a} &= c, \end{aligned} \tag{26}$$

giving $r^2 = \frac{c}{a}$ and hence we have finally, $r = \sqrt{\frac{c}{a}}$ and $\theta = \arccos\frac{-b}{2\sqrt{ac}}$ and we have therefore found the two solutions given by

$$x = r\mathrm{e}^{\pm\iota_2\theta}, \tag{27}$$

which is in fact equivalent to the standard quadratic formula, but only employing real numbers. Also we can view the quadratic equation as a vector equation. Multiplying the quadratic equation through by e_1 we find

$$\begin{aligned} -ar^2(\cos\theta + \iota\sin\theta)^2 e_1 - br(\cos\theta + \iota\sin\theta)e_1 &= ce_1 \\ -ar^2\mathbf{w}e_1\mathbf{w} - br\mathbf{w} &= ce_1 \end{aligned} \tag{28}$$

where $\mathbf{w} = \cos\theta e_1 - \sin\theta e_2$ is an unknown unit vector, with $\mathbf{w}^2 = 1$. Multiplying by $\mathbf{w}e_1$ we find

$$-ar^2\mathbf{w} - bre_1 = ce_1\mathbf{w}e_1$$

a vector equation, which we can solve by equating e_1 and e_2 components returning the same solution as previously.

In summary the appearance of imaginary numbers in the solution of quadratic equations, can be seen as solutions involving rotations in the plane. This idea can therefore be generalized to allow rotations in three-space using three-space rotors. This will be equivalent to allowing x to be a quaternion, or represented in GA by defining $x = \mathrm{e}^{\iota\hat{u}\theta}$, where $\hat{u}$ is a unit vector. We find that the same solution is obtained as previously of $r = \sqrt{\frac{c}{a}}$ and $\theta = \arccos\frac{-b}{2\sqrt{ac}}$, which shows that the axis of rotation $\hat{u}$ has no effect, showing we have an infinite number of solutions. For the two-space case we were essentially assuming that we were rotating about the e_3 axis, clearly then, any other axis will also provide a solution to the quadratic. We thus have provided a geometric interpretation of the solution of quadratics in two-space or three-space, and thus the solutions of the quadratics are always real, representing rotations. Other extensions now present themselves, including, letting x become a full multivector.

H. The Dirac equation

We find using GA, that we can write the free Dirac equation in real three-space as

$$\Box F = -\frac{mc}{\hbar}F^*\iota e_3 \tag{29}$$

where $F = a + \mathbf{E} + \iota\mathbf{B} + \iota b$ and $F^* = e_0 F e_0 = a - \mathbf{E} + \iota\mathbf{B} - \iota b$, space conjugation.

Thus the Dirac equation using three-space GA simply becomes an extension of the free Maxwell equation shown in Eq. (8), with the addition of a mass (m) term and expansion of the field F to a full multivector as shown. This is clearly a dramatic simplification of the standard Dirac equation, which is normally considered embedded in four-dimensional space time employing 4×4 complex matrices, as shown in Appendix B. Hence we can recognize another over-complication in current mathematical formalisms with this example, in that the non-commuting property of matrices is used as a substitute for the much simpler non-commuting real basis vectors of GA. Also, with the GA form of the Dirac equation, we can

see that it describes a multivector field, given by Eq. (9), over real three-space, that is, at each point in three-space we have a multivector valued field defined.

II. CONCLUSION

In this paper we ask the question about the correct algebraic representation of three-space, and we conclude that GA provides the simplest and most natural formalism. Geometric algebra that uses non-commuting Cartesian coordinates, subsumes the rotational algebra of complex numbers and quaternions into a simple algebraic system over the real field. We claim that Clifford's GA therefore has general applicability, which we illustrate with the two examples of Maxwell's equations and the Dirac equation, both being written as single equations in real three dimensional space. This approach thus has the advantage that all algebraic entities have an immediate geometric representation in real three-space.

We argue that the current system of vectors taught to students, which lacks an inverse operation with regards to multiplication, should be replaced with the system of vectors described in GA, in which multiplication and division of vectors becomes a natural algebraic process. We show in Eq. (1), how the geometric product described by Clifford is a natural extension of the distributive law for brackets incorporating non-commutivity. The non-commutivity is required for three-space, simply because rotations in three dimensions are non-commuting.

We also show more generally how Clifford's union of algebra and geometry in three-space allows the viewing of a multivector, as shown in Eq. (9) as a generalized type of number, useful in representing different physical quantities, amenable to the basic operations of addition, subtraction, multiplication, division and square root. We showed how a general quadratic equation can be solved without recourse to complex numbers, giving the solutions geometric meaning as rotations in the plane, which will be beneficial to students learning algebra.

We highlight two foundational errors in the development of the current mathematical formalisms. First, the separation of the dot and cross products into two separate products, rather than considering them as a single unified product, and second, the use of the Pauli and Dirac complex matrices, as a way to implement non-commutivity, in place of the much simpler orthonormal real basis vectors of GA.

III. APPENDIX

A. Application to Electromagnetism

In order to gain a wider comparison between the conventional vector calculus form of electromagnetism and the GA form, we list below various expressions in both formalisms including the tensor formalism. The GA formalism uses a single variable F to hold the electric field strength consisting of a vector for $\mathbf{E}$ and a pseudo vector for the $\mathbf{B}$ field, combining the four Maxwell equations into a single equation. When calculating the energy in the fields or the invariants, there is also the advantage of simplicity, in that only a single equation is required to produce the energy and momentum or the two invariants, as shown in Table I. The energy and momentum expression is in a form similar to kinetic energy, that is $\frac{1}{2}\epsilon_0|F|^2$. In the other cases listed in the table, we can see in all cases GA provides more compact expressions.

B. Dirac equation

Schrödinger developed a non-relativistic wave equation in 1925 which was generalized to include spin by Pauli. Dirac then sought a relativistic formulation and succeeded in 1928 in producing the explicitly covariant Dirac equation

$$- i\hbar\gamma^\mu\partial_\mu\psi + mc\psi = 0, \tag{30}$$

which uses the Einstein summation convention, where

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \tag{31}$$

and

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{32}$$

As discussed, if the gamma matrices are considered as basis vectors, we would then have the anti commutator $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\gamma^\mu \cdot \gamma^\nu$, as expected for a set of orthonormal

EM theory	Gibbs' vector calculus	Tensors	GA
Fields	$\mathbf{E}$,$c\mathbf{B}$	$F^{\alpha\beta}$	$\mathbf{E}+\iota c\mathbf{B}$
EM equations	$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon}$, $\nabla\times\mathbf{E}+\partial_t\mathbf{B}=0$	$F^{\alpha\beta}_{,\alpha}=c\mu_0 J^{\beta}$	$\Box F=c\mu_0 J$
	$\nabla\times\mathbf{B}-\frac{1}{c^2}\partial_t\mathbf{E}=\mu_0\mathbf{J}$, $\nabla\cdot\mathbf{B}=0$	$\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta,\gamma}=0$	
Charge conservation	$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0$	$J^{\alpha}_{,\alpha}=0$	$\Box\cdot J=0$
Energy in fields U	$\frac{1}{2}\epsilon(\mathbf{E}^2+c^2\mathbf{B}^2)$	$\epsilon(F^{0\gamma}F^{0}_{\gamma}+\frac{1}{4}F_{\mu\nu}F^{\mu\nu})$	$-\frac{1}{2}\epsilon_0\|F\|^2$
Poynting vector $\mathbf{S}/c$	$\frac{1}{\mu_0 c}(\mathbf{E}\times\mathbf{B})$	$\epsilon F^{0\gamma}F^{j}_{\gamma}$	
Invariants	$c^2\mathbf{B}^2-\mathbf{E}^2$	$\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}$	F^2
	$\frac{2}{c}\mathbf{B}\cdot\mathbf{E}$	$\frac{1}{4}\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta}$	
Minkowski Force	$\mathbf{K}=\gamma q(\mathbf{E}+\mathbf{v}\times\mathbf{B}), cK^0=q\gamma\mathbf{v}\cdot\mathbf{E}$	$cK^{\mu}=-qv_{\nu}F^{\mu\nu}$	$cK=-qvF$
Potential function A	$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}$, $\mathbf{B}=\nabla\times\mathbf{A}$	$F_{\alpha\beta}=A_{\alpha,\beta}-A_{\beta,\alpha}$	$F=\Box A$
Lorenz gauge	$\lambda=\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}=0$	$A^{\alpha}_{,\alpha}=0$	$\Box\cdot A=0$
Potential form	$\Box^2\mathbf{A}=-\mu_0\mathbf{J}$, $\Box^2 V=-\frac{\rho}{\epsilon_0}$	$\Box^2 A^{\mu}=-c\mu_0 J^{\mu}$	$\Box^2 A=c\mu_0 J$

TABLE I: Comparison of mathematical formalisms, showing relative simplicity of mathematical expressions in GA. Note: The blanks in several rows in the column for GA are because the relevant equation is automatically part of the equation above. For example, there is a blank row below the F^2 entry for GA, because the separate scalar and pseudoscalar components of this expression produce the two invariants in the vector column.

basis vectors. Hence, the opinion of many people, that Dirac indirectly rediscovered Clifford's geometric algebra with its anti-commuting basis vectors.

Dirac's complex, four-space equation using 4×4 complex matrices, can be compared with the real three-space version, shown in Eq. (29).

C. Operators on multivectors

1. *The inverse operation*

Given a general three-space multivector

$$M=a+\mathbf{v}+\iota\mathbf{w}+\iota b \tag{33}$$

we seek an inverse multivector, so that the division operation can be achieved. Firstly, we find that

$$Z = MM^\dagger = (a + \mathbf{v} + \iota\mathbf{w} + \iota b)(a - \mathbf{v} - \iota\mathbf{w} + \iota b) = a^2 - v^2 + w^2 - b^2 + 2\iota(ab - \mathbf{v}\cdot\mathbf{w}). \quad (34)$$

However this produces a number with a scalar and trivector components, we therefore need to multiply by its conjugate in order to create a scalar, giving

$$d = Z\tilde{Z} = (a^2 - v^2 + w^2 - b^2)^2 + 4(ab - \mathbf{v}\cdot\mathbf{w})^2. \quad (35)$$

Therefore we find the inverse

$$\begin{aligned} M^{-1} &= M^\dagger \tilde{Z}/d \quad (36) \\ &= (a - \mathbf{v} - \iota\mathbf{w} + \iota b)(a^2 - v^2 + w^2 - b^2 - 2\iota(ab - \mathbf{v}\cdot\mathbf{w})/d. \end{aligned}$$

This will fail to produce an inverse if $d = 0$, that by inspection requires both

$$a^2 + w^2 = v^2 + b^2 \text{ and } ab = \mathbf{v}\cdot\mathbf{w}. \quad (37)$$

2. The square root

To find the square root of a multivector M, we need to find a multivector N, such that

$$N^2 = M \quad (38)$$

where

$$M = a + \mathbf{v} + \iota\mathbf{w} + \iota b \quad (39)$$

is a general multivector. In order to streamline calculations we replace

$$M = (a + \iota b)(1 + \mathbf{m} + \iota\mathbf{n}) \quad (40)$$

where by inspection $\mathbf{m} = \frac{a\mathbf{v}+b\mathbf{w}}{h_0}$, $\mathbf{n} = \frac{a\mathbf{w}-b\mathbf{v}}{h_0}$ and $h_0 = a^2 + b^2$. Clearly this result is not valid if $h_0 = 0$, which would imply that $a = b = 0$ implying a pure vector and bivector, which can be found to have the solution $\sinh\theta = -\frac{q_0}{p_0} \pm \sqrt{\frac{q_0^2}{p_0^2} + 1}$, where $q_0 = \frac{v^2-w^2}{4}$ and $p_0 = \frac{1}{2}\mathbf{v}\cdot\mathbf{w}$. A subclass of solutions is also produced if $p_0 = 0$, giving $c = d = q_0^{1/4}$. The case with $q_0 = 0$ in fact appears to have no solution, that is the multivector $M = \mathbf{u} + \iota\mathbf{u}^\perp$, has no square

root, although it can be approximated arbitrarily closely with $N = \epsilon + \frac{\mathbf{u}}{2\epsilon} + \frac{i\mathbf{u}^{\perp}}{2\epsilon} + \epsilon$, giving $M = \mathbf{u} + \iota\mathbf{u}^{\perp} + 2\iota\epsilon^2$. So for cases with $h_0 \neq 0$ we seek a multivector such that

$$\begin{aligned} &= (c + \mathbf{x} + \iota\mathbf{y} + \iota d)(c + \mathbf{x} + \iota\mathbf{y} + \iota d) \\ &= (c^2 + x^2 - y^2 - d^2) + 2(c\mathbf{x} - d\mathbf{y}) + 2\iota(c\mathbf{y} + d\mathbf{x}) + 2\iota(cd + \mathbf{x}\cdot\mathbf{y}) \\ &= 1 + \mathbf{m} + \iota\mathbf{n}. \end{aligned} \tag{41}$$

We can see that now we have constructed a simplified problem, of the square root of a multivector with $b = 0$, and $a = 1$, but from which we can reconstruct the full square root. Inspecting the six linear equations formed from the vector and trivector components we can see that we require

$$\mathbf{x} = \frac{d\mathbf{n} + c\mathbf{m}}{2h} \quad , \quad \mathbf{y} = \frac{c\mathbf{n} - d\mathbf{m}}{2h} \tag{42}$$

where $h = c^2 + d^2$. For the case $h = 0$, which implies $c = d = 0$, also implies $N = a + \iota b$, which has a known square root as for a complex number, calculated later in the appendix, and so for more general cases we can assume $h > 0$. We now need to just find c and d from the two remaining equations in Eq. (41) of

$$\begin{aligned} c^2 - d^2 + \frac{(m^2 - n^2)(c^2 - d^2)}{4h^2} + \frac{4cd\mathbf{m}\cdot\mathbf{n}}{4h^2} &= 1 \\ cd - \frac{cd(m^2 - n^2) + (c^2 - d^2)\mathbf{m}\cdot\mathbf{n}}{4h^2} &= 0 \end{aligned} \tag{43}$$

and after multiplying through by h^2 and making the replacement $q = \frac{m^2 - n^2}{4}$ and $p = \frac{1}{2}\mathbf{m}\cdot\mathbf{n}$, we find

$$\begin{aligned} (c^2 - d^2)(c^2 + d^2)^2 + q(c^2 - d^2) + 2cdp &= (c^2 + d^2)^2 \\ 2cd(c^2 + d^2)^2 - 2cdq + (c^2 - d^2)p &= 0. \end{aligned} \tag{44}$$

We have difficult simultaneous polynomials in c and d, and so it is now convenient to make the substitution

$$c = r\cosh\frac{\theta}{2} \quad , \quad d = r\sinh\frac{\theta}{2}, \tag{45}$$

finding

$$
\begin{aligned}
r^4\cosh^2\theta + q + p\sinh\theta &= r^2\cosh^2\theta \\
r^4\sinh\theta\cosh^2\theta - \sinh\theta q + p &= 0.
\end{aligned}
\tag{46}
$$

From the second equation in Eq. (46), we find

$$
r = \pm\left(\frac{q - p\,\mathrm{cosech}\theta}{\cosh^2\theta}\right)^{\frac{1}{4}}. \tag{47}
$$

We can see that either sign for r satisfies the two equations of Eq. (46) although this will only flip the overall sign of the square root, as expected, giving us a square root for the multivector of either sign. The term under the fourth root, $q - p\,\mathrm{cosech}\theta$ in fact remains non-negative, and hence r remains real. On substitution back into the first equation in Eq. (46) we find the trigonometric equation,

$$
2q - p\,\mathrm{cosech}\theta(1-\sinh^2\theta) = \cosh\theta\sqrt{q - p\,\mathrm{cosech}\theta} \tag{48}
$$

and substituting $x = \sinh\theta$ we find the quartic equation in x

$$
(q-p^2)x^4 - p(1+4q)x^3 + \left(2p^2 - q(4q-1)\right)x^2 + p(4q-1)x - p^2 = 0 \tag{49}
$$

which has a solution

$$
x = \frac{p(1+4q+s) \pm u}{4(q-p^2)}, \tag{50}
$$

where $s = +\sqrt{(4q-1)^2 + 16p^2}$ and $u = \sqrt{2}\sqrt{16q^3 + 12qp^2 + p^2 - 4q^2 + s(p^2+4q^2)}$. We can see that s remains real, but if we allow a negative square root of s inside the equation for u, then we will produce an imaginary term and hence we need to maintain a positive square root for s which leaves just the two real solutions to the quartic as shown in Eq. (50). Hence we have two distinct $\theta = \mathrm{arcsinh}x$, which imply two distinct square roots, ignoring signs. We can see that there is a special case $q = p^2$ that will reduce Eq. (49) to a cubic,

$$
p(1+4p^2)x^3 + p^2(4p^2-3)x^2 - p(4p^2-1)x + p^2 = 0, \tag{51}
$$

which has the single real solution $x = -p$, giving a single root with $\theta = -\mathrm{arcsinh}p$. If $p = 0$, then the cubic fails and we find two cases dependent on q, with firstly $\cosh\theta = 2\sqrt{q}$, requiring $q \geq \frac{1}{4}$ and secondly $\theta = 0$ with $r = c$, where $c = \sqrt{\frac{1+\sqrt{1-4q}}{2}}$, valid for $q \leq \frac{1}{4}$.

The square root of $a + \iota b$ is easily found to be $f + \iota g = \pm\frac{1}{\sqrt{2}}(\sqrt{a + \sqrt{h_0}} + \iota\sqrt{-a + \sqrt{h_0}})$ Hence for M given by Eq. (39), we find the square root

$$M^{\frac{1}{2}} = \pm(c + \mathbf{x} + \iota\mathbf{y} + \iota d)\,(f + \iota g)\,, \tag{52}$$

where c, d given by Eq. (45) and $\mathbf{x}, \mathbf{y}$ given by Eq. (42). Because multivectors are associative we can now find the rational powers $M^{k/2^t}$, where k, t are integers, which can approximate to any fractional power. We find that the square root generally exists, though in some cases there is a double root (ignoring signs) and for the special case mentioned we can only approximate the root arbitrarily closely.

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